Quasi-invariant flow generated by Stratonovich SDE with BV drift coefficients

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Abstract

We generalize the results of Ambrosio [Invent. Math. 158 (2004), 227–260] on the existence, uniqueness and stability of regular Lagrangian flows of ordinary differential equations to Stratonovich stochastic differential equations with BV drift coefficients. Then we construct an explicit solution to the corresponding stochastic transport equation in terms of the stochastic flow. The approximate differentiability of the flow is also studied when the drift is a Sobolev vector field.

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1 Introduction

Let $A_0, A_1, \cdots, A_m$ be vector fields on $\mathbb{R}^d$ and $w_t = (w_t^1, \cdots, w_t^m)$ an m-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the Stratonovich stochastic differential equation (abbreviated as SDE)

$$\mathrm{d}X_t = \sum_{i=1}^m A_i(X_t) \circ \mathrm{d}w_t^i + A_0(X_t)\,\mathrm{d}t, \quad X_0 = x. \tag{1.1}$$

It is well known that if $A_i \in C_b^{2+\delta}(\mathbb{R}^d, \mathbb{R}^d)\,(i = 1, \cdots, m)$ and $A_0 \in C_b^{1+\delta}(\mathbb{R}^d, \mathbb{R}^d)$, then the above equation has a unique solution which defines a stochastic flow of C^1-diffeomorphisms on $\mathbb{R}^d$. Here $C_b^{1+\delta}$ means that the vector field A_0 and its first order derivatives are bounded, and the derivatives are Hölder continuous of order $\delta > 0$. These conditions on the boundedness of the vector fields and their derivatives were relaxed in [18], by allowing the local Lipschitz constants on the balls centered at the origin to grow as fast as the logarithmic function. In the case $\delta = 0$, it is proved in [12] that under the same growth conditions, (1.1) still gives rise to a flow of homeomorphisms on $\mathbb{R}^d$. This result is generalized in [13] to the case where the drift coefficient A_0 satisfies only the general Osgood condition, at the price of the $C_b^{3+\delta}$ regularity for the diffusion coefficients; if in addition the distributional divergence of A_0 exists and is bounded, then the Lebesgue measure is quasi-invariant under the action of the stochastic flow of homeomorphisms (cf. [19]).

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On the other hand, the ordinary differential equation (abbreviated as ODE)

$$\mathrm{d}X_t = A_0(X_t)\,\mathrm{d}t, \quad X_0 = x \tag{1.2}$$

with Sobolev or even BV coefficient has been studied intensively in the last three decades. Here A_0 can be a time-dependent vector field. The existence of quasi-invariant flow of measurable maps associated to a vector field A_0 with Sobolev regularity was first studied by Cruzeiro [8]. A thorough treatment was carried out by DiPerna and Lions in the celebrated paper [9], where the authors deduced the existence and uniqueness of flows generated by (1.2) from the well posedness of the corresponding transport equations. Similar results were obtained in [6] by taking the standard Gaussian measure as the reference measure. Ambrosio [1] generalized the results to the case where A_0 has only BV regularity by considering the continuity equation. S. Fang [11] gave a short introduction to the theories mentioned above. The extension of these results to the infinite dimensional Wiener space have been done in [3, 14]. Using the local maximal function, Crippa and De Lellis obtained in [7] some new estimates which allow them to give a direct proof of the existence and uniqueness of the DiPerna-Lions flow.

Inspired by these studies of ODE, there have been several attempts to solve the SDE with Sobolev coefficients. Following the method in [7], X. Zhang [23] showed the existence and uniqueness of the stochastic flow of measurable maps generated by Itô SDE with Sobolev coefficients, provided that the derivatives of the diffusion vector fields are bounded. The SDE with BV drift vector field was also considered in this paper, but the diffusion coefficients were assumed to be constant. In [15], the authors took the standard Gaussian measure as the reference measure and proved a priori estimate on the L^p norm of the density of the flow, which enabled them to construct the unique flow associated to (1.1), provided that the gradients of the diffusion coefficients and the divergences with respect to the Gaussian measure are exponentially integrable. In the recent work [24], X. Zhang studied the Stratonovich SDE with drift coefficient belonging to $W^{1,1}_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, and he also provided a Freidlin-Wentzell type large deviation estimate for the stochastic flow.

In the present paper we consider the Stratonovich SDE (1.1) with BV drift vector field. Our method is based on Ocone-Pardoux's decomposition [21] of the flow generated by (1.1) into the stochastic flow of the diffusion part, and a flow associated to random ODEs whose driving vector field is a transformation of the drift coefficient A_0 by the stochastic flow. This approach was applied in [13] to deal with the Stratonovich SDE with drift satisfying the general Osgood condition. Our main result can be stated as follows ($\mathcal{L}_d$ is the Lebesgue measure on $\mathbb{R}^d$).

Theorem 1.1. *Assume that $A_1, \cdots, A_m \in C_b^{3+\delta}(\mathbb{R}^d,\mathbb{R}^d)$, and the drift A_0 satisfies*

(1) *A_0 has sublinear growth, i.e. $|A_0(x)| \le C(1+|x|^{1-\varepsilon_0})$, $x\in\mathbb{R}^d$;*

(2) *$A_0 \in \mathrm{BV}_{loc}(\mathbb{R}^d,\mathbb{R}^d)$;*

(3) *the divergence $D\cdot A_0 \ll \mathcal{L}_d$ with locally bounded density function $\mathrm{div}(A_0)$.*

Then the equation (1.1) generates a unique stochastic flow of measurable maps $X_t : \mathbb{R}^d \to \mathbb{R}^d$ which leaves the Lebesgue measure $\mathcal{L}_d$ quasi-invariant.

This result will be proved in Section 3. Note that the sublinear growth of A_0 ensures that the vector field $\tilde{A}_0$ defined in (3.1) has similar growth (Lemma 3.1), which in turn implies the classical growth estimates on the solution of the ODE. The sublinear growth of A_0 also allows us to assume only the local boundedness of the divergence $\mathrm{div}(A_0)$ (as in (ii) of Theorem 6.2 in [1]), compared to the global boundedness required in [2, 9]. The starting point of the proof of Theorem 1.1 is the relationship between the distributional derivative (a Radon measure in

the present case) of the drift A_0 and that of the transformed vector field $\tilde{A}_0$ defined in (3.1). This will be done in Lemma 2.2, where we also show that if the divergence $D\cdot A_0$ is absolutely continuous with respect to $\mathcal{L}_d$, then so is $D\cdot\tilde{A}_0$, and they are related to each other by a quite simple equality.

We study in Section 4 the stability of the solutions to (1.1) when a sequence of vector fields A_0^n converge in some sense to A_0. For this purpose, denote by $B(R)$ the ball centered at the origin with radius R.

Theorem 1.2. *Assume the conditions of Theorem 1.1. Let $A_0^n:\mathbb{R}^d\to\mathbb{R}^d$ be vector fields satisfying*

(1) *there is $C>0$ and $\varepsilon_0\in(0,1)$, such that $\sup_{n\geq 1}|A_0^n(x)|\leq C(1+|x|^{1-\varepsilon_0})$, $x\in\mathbb{R}^d$;*

(2) *A_0^n converges to A_0 in $L^1_{loc}(\mathbb{R}^d,\mathbb{R}^d)$;*

(3) *for any $n\geq 1$, ∇A_0^n is locally bounded;*

(4) *for any $R>0$, $\sup_{n\geq 1}\|\mathrm{div}(A_0^n)\|_{L^\infty(B(R))}<+\infty$.*

Let X_t^n be the flow associated to (1.1) with A_0 being replaced by A_0^n. Then for any $p>1$ and T, $R>0$, the following convergence holds almost surely and in $L^p(\Omega,\mathbb{P})$:

$$\lim_{n\to\infty}\int_{B(R)}\sup_{t\in[0,T]}|X_t^n(x)-X_t(x)|\,\mathrm{d}x=0.$$

It is well known that when the coefficients are smooth, the solution to the corresponding stochastic transport equation can be explicitly expressed in terms of the flow generated by (1.1), see [13] Theorem 5.1 for the case where A_0 satisfies the general Osgood condition. As an application of the above stability result, we will show that similar representation still holds even in the situation of BV drift.

Finally in Section 5 we consider slightly more regular drift coefficient $A_0\in W^{1,1}_{loc}$, showing that almost surely, the stochastic flow X_t is approximately differentiable on $\mathbb{R}^d$. This generalizes the results in [4, 7] to the stochastic context.

2 Preparations and known results of ODE

In this section we give some preliminary results needed in the subsequent sections. Here is the definition of BV functions.

Definition 2.1. *A locally integrable function $b:\mathbb{R}^d\to\mathbb{R}^m$ is said to be of class BV_{loc} if there is a $\mathbb{R}^{m\times d}$-valued Radon measure $Db=(D_ib^j)_{1\leq i\leq d,\,1\leq j\leq m}$ such that*

$$\int_{\mathbb{R}^d}\psi\,\mathrm{d}(D_ib^j)=-\int_{\mathbb{R}^d}b^j\partial_i\psi\,\mathrm{d}x,\quad \textit{for all }\psi\in C_c^1(\mathbb{R}^d).$$

If $b\in C^1$, then we still denote by Db the function ∇b on $\mathbb{R}^d$. If $m=d$, we denote by $D\cdot b=\mathrm{tr}(Db)=\sum_{i=1}^d D_ib^i$ the divergence of the BV vector field b, which is again a Radon measure on $\mathbb{R}^d$. In the following $\det(\cdot)$ means the determinant of a matrix. For a measurable map $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ and a Radon measure μ on $\mathbb{R}^d$, $\varphi_{\#}\mu$ denotes the push forward of the measure μ by φ (or the "distribution" of φ under μ). Now we prove

Lemma 2.2. *Let $b:\mathbb{R}^d\to\mathbb{R}^d$ be a BV_{loc} vector field and $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ a C^2-diffeomorphism, then*

(1) *the composition $b\circ\varphi$ is still a BV_{loc} vector field and*

$$D(b\circ\varphi)=|\det(J_\varphi)|^{-1}J_\varphi^*\,[(\varphi^{-1})_\# Db],$$

where J_φ is the Jacobi matrix of φ and J_φ^* is its transpose, $J_\varphi^*\,[(\varphi^{-1})_\# Db]$ is the matrix product of J_φ^* and $(\varphi^{-1})_\# Db$;

(2) *if the divergence $D\cdot b\ll\mathcal{L}_d$ with density function $\mathrm{div}(b)$, then*

$$D\cdot[J_\varphi^{-1}(b\circ\varphi)]=\langle\mathrm{div}(J_\varphi^{-1}),b\circ\varphi\rangle+\mathrm{div}(b)\circ\varphi,$$

where $\mathrm{div}(J_\varphi^{-1})$ is a vector field whose components are the divergences of the column vectors of J_φ^{-1}.

Proof. (1) For every $i\in\{1,\cdots,d\}$, we only have to show that $b^i\circ\varphi$ is a BV function. By Theorems 2 and 3 in Section 5.2 of [10], there exists a sequence of functions $b_n^i\in\mathrm{BV}_{loc}\cap C^\infty(\mathbb{R}^d)$ such that $b_n^i\to b^i$ in $L^1_{loc}(\mathbb{R}^d)$ and the vector valued measures Db_n^i converges weakly to Db^i as $n\to\infty$. Note that the composition $b_n^i\circ\varphi$ belongs to C^2, hence for any $\psi\in C_c^\infty(\mathbb{R}^d)$, by the integration by parts formula,

$$\int_{\mathbb{R}^d}\psi\,[D(b_n^i\circ\varphi)]\,\mathrm{d}x=-\int_{\mathbb{R}^d}(b_n^i\circ\varphi)\nabla\psi\,\mathrm{d}x. \tag{2.1}$$

We have by the chain rule, $D(b_n^i\circ\varphi)=J_\varphi^*\,[(Db_n^i)\circ\varphi]$. It follows from the formula of changing variables that

$$\begin{aligned}\int_{\mathbb{R}^d}\psi\,[D(b_n^i\circ\varphi)]\,\mathrm{d}x&=\int_{\mathbb{R}^d}\psi J_\varphi^*\,[(Db_n^i)\circ\varphi]\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}[(\psi J_\varphi^*)\circ(\varphi^{-1})](Db_n^i)|\det(J_{\varphi^{-1}})|\,\mathrm{d}x\\&\to\int_{\mathbb{R}^d}[(\psi J_\varphi^*)\circ(\varphi^{-1})]\cdot|\det(J_{\varphi^{-1}})|\,\mathrm{d}(Db^i)\end{aligned}$$

as $n\to\infty$, due to the weak convergence of $(Db_n^i)\,\mathrm{d}x$ to Db^i. Now we have

$$\begin{aligned}\int_{\mathbb{R}^d}[(\psi J_\varphi^*)\circ(\varphi^{-1})]\cdot|\det(J_{\varphi^{-1}})|\,\mathrm{d}(Db^i)&=\int_{\mathbb{R}^d}\psi|\det(J_{\varphi^{-1}})\circ\varphi|J_\varphi^*\,\mathrm{d}[(\varphi^{-1})_\# Db^i]\\&=\int_{\mathbb{R}^d}\psi|\det(J_\varphi)|^{-1}J_\varphi^*\,\mathrm{d}[(\varphi^{-1})_\# Db^i],\end{aligned}$$

where the last equality follows from $J_{\varphi^{-1}}\circ\varphi=(J_\varphi)^{-1}$. Therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\psi\,[D(b_n^i\circ\varphi)]\,\mathrm{d}x=\int_{\mathbb{R}^d}\psi|\det(J_\varphi)|^{-1}J_\varphi^*\,\mathrm{d}[(\varphi^{-1})_\# Db^i]. \tag{2.2}$$

Now we consider the limit of the right hand side of (2.1). Again by changing variables,

$$-\int_{\mathbb{R}^d}(b_n^i\circ\varphi)\nabla\psi\,\mathrm{d}x=-\int_{\mathbb{R}^d}b_n^i\cdot[(\nabla\psi)\circ\varphi^{-1}]\cdot|\det(J_{\varphi^{-1}})|\,\mathrm{d}x$$

which converges to

$$-\int_{\mathbb{R}^d}b^i\cdot[(\nabla\psi)\circ\varphi^{-1}]\cdot|\det(J_{\varphi^{-1}})|\,\mathrm{d}x=-\int_{\mathbb{R}^d}(b^i\circ\varphi)\nabla\psi\,\mathrm{d}x.$$

This combines with (2.1) and (2.2) leads to

$$\int_{\mathbb{R}^d} \psi |\det(J_\varphi)|^{-1} J_\varphi^* \,\mathrm{d}[(\varphi^{-1})_\# Db^i] = -\int_{\mathbb{R}^d} (b^i \circ \varphi) \nabla \psi \,\mathrm{d}x,$$

which means that $b^i \circ \varphi$ is a BV function and

$$D(b^i \circ \varphi) = |\det(J_\varphi)|^{-1} J_\varphi^* \,[(\varphi^{-1})_\# Db^i].$$

(2) By the chain rule and (1), it is easy to know that $J_\varphi^{-1}(b \circ \varphi)$ is a BV_{loc} vector field, and

$$D \cdot [J_\varphi^{-1}(b \circ \varphi)] = \langle \mathrm{div}(J_\varphi^{-1}), b \circ \varphi \rangle + \langle J_\varphi^{-1}, D(b \circ \varphi) \rangle, \tag{2.3}$$

where the second $\langle \cdot, \cdot \rangle$ is the inner product of matrices regarded as elements in $\mathbb{R}^{d \times d}$. By the expression in (1), we have

$$\langle J_\varphi^{-1}, D(b \circ \varphi) \rangle = |\det(J_\varphi)|^{-1} \mathrm{tr}[(\varphi^{-1})_\# Db] = |\det(J_\varphi)|^{-1} (\varphi^{-1})_\# \mathrm{tr}(Db). \tag{2.4}$$

Since $\mathrm{tr}(Db) = D \cdot b = \mathrm{div}(b) \mathcal{L}_d$, for any $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \psi \,\mathrm{d}[(\varphi^{-1})_\# \mathrm{tr}(Db)] &= \int_{\mathbb{R}^d} (\psi \circ \varphi^{-1}) \,\mathrm{d}[\mathrm{tr}(Db)] = \int_{\mathbb{R}^d} (\psi \circ \varphi^{-1}) \cdot \mathrm{div}(b) \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \psi \cdot [\mathrm{div}(b) \circ \varphi] \cdot |\det(J_\varphi)| \,\mathrm{d}x.\end{aligned}$$

Therefore $(\varphi^{-1})_\# \mathrm{tr}(Db) = [\mathrm{div}(b) \circ \varphi] \cdot |\det(J_\varphi)|$. Combining this equality with (2.3) and (2.4), we complete the proof. □

The next technical result will be used in Section 4.

Lemma 2.3. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be a C^2-diffeomorphism and $\tilde\rho := \frac{\mathrm{d}(\varphi_\# \mathcal{L}_d)}{\mathrm{d}\mathcal{L}_d}$ the Radon-Nikodym density function. Then we have*

$$(\tilde\rho^{-1} \nabla \tilde\rho) \circ \varphi = \mathrm{div}\big(J_\varphi^{-1}\big).$$

Proof. It is well known that $\tilde\rho \circ \varphi = \big|\det\big(J_\varphi^{-1}\big)\big|$. Since φ is a C^2-diffeomorphism of $\mathbb{R}^d$, the function $\mathbb{R}^d \ni x \mapsto \det\big(J_\varphi^{-1}\big)(x)$ does not change sign. Without loss of generality, we may assume that $\det\big(J_\varphi^{-1}\big) > 0$ on the whole $\mathbb{R}^d$, thus $\tilde\rho \circ \varphi = \det\big(J_\varphi^{-1}\big)$. As $\nabla(\tilde\rho \circ \varphi) = J_\varphi^* [(\nabla \tilde\rho) \circ \varphi]$, we have

$$(\nabla \tilde\rho) \circ \varphi = (J_\varphi^{-1})^* \nabla (\tilde\rho \circ \varphi) = (J_\varphi^{-1})^* \nabla \det\big(J_\varphi^{-1}\big).$$

Note that $\nabla \det\big(J_\varphi^{-1}\big) = -[\det(J_\varphi)]^{-2} \nabla \det(J_\varphi)$, so the equality that we should prove is

$$\nabla \det(J_\varphi) = -\det(J_\varphi) J_\varphi^* \,\mathrm{div}\big(J_\varphi^{-1}\big).$$

For simplification of the notations we write $J = J_\varphi$ and $K = J_\varphi^{-1}$, then $J_{ij} = \partial_j \varphi^i$, $1 \le i, j \le d$. We consider the determinant $\det(\cdot)$ as a smooth function defined on $\mathbb{R}^{d \times d}$. By the chain rule and Jacobi's formula (see [20] Part Three, Section 8.3), we have for any $l \in \{1, \cdots, d\}$,

$$\partial_l \det(J) = \sum_{i,j=1}^d \frac{\partial \det}{\partial x_{ij}}(J) \cdot \partial_l J_{ij} = \sum_{i,j=1}^d \det(J) K_{ji} \partial_{lj} \varphi^i. \tag{2.5}$$

For any $1 \le j \le d$, it holds $\delta_{jl} = \sum_{i=1}^d K_{ji} J_{il} = \sum_{i=1}^d K_{ji} \partial_l \varphi^i$, therefore

$$\sum_{i=1}^d (\partial_j K_{ji})(\partial_l \varphi^i) + \sum_{i=1}^d K_{ji}(\partial_{jl}\varphi^i) = 0,$$

that is, $\sum_{i=1}^d K_{ji}(\partial_{jl}\varphi^i) = -\sum_{i=1}^d (\partial_j K_{ji})(\partial_l \varphi^i)$. Combining this with (2.5) leads to

$$\begin{aligned}
\partial_l \det(J) &= -\det(J) \sum_{j=1}^d \sum_{i=1}^d (\partial_j K_{ji})(\partial_l \varphi^i) = -\det(J) \sum_{i=1}^d (\partial_l \varphi^i) \sum_{j=1}^d (\partial_j K_{ji}) \\
&= -\det(J) \sum_{i=1}^d J_{il} \mathrm{div}(K_{\cdot i}) = -\det(J)(J^* \mathrm{div}(K))_l.
\end{aligned}$$

The proof is complete. □

Now we recall the definition of the regular Lagrangian flow associated to a time-dependent vector field b_t (see [2, 7]).

Definition 2.4. *Let $b \in L^1_{loc}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. We call a map $Y : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ a regular Lagrangian flow for the vector field b if*

(1) *for a.e. $x \in \mathbb{R}^d$, the map $[0,T] \ni t \to Y_t(x)$ is an absolutely continuous integral solution of*

$$\mathrm{d}Y_t = b_t(Y_t)\,\mathrm{d}t, \quad Y_0 = x; \tag{2.6}$$

(2) *$(Y_t)_\# \mathcal{L}_d \ll \mathcal{L}_d$ for all $t \in [0,T]$.*

Note that this definition is slightly different from that in [7]: in condition (2) we do not require that $(Y_t)_\# \mathcal{L}_d$ is dominated by $C\mathcal{L}_d$ on the whole $\mathbb{R}^d$, where $C > 0$ is a constant.

Given a measurable map $Y : \mathbb{R}^d \to \mathbb{R}^d$, we say that $Z : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable inverse map of Y if Z is measurable and for a.e. $x \in \mathbb{R}^d$, $x = Y(Z(x)) = Z(Y(x))$. We often denote by Y^{-1} the measurable inverse map of Y (see [23] Lemma 3.4 for a characterization of this notion). In the following theorem we summarize the results concerning the existence and uniqueness of regular Lagrangian flow generated by a BV vector field.

Theorem 2.5. *Let $b_t : \mathbb{R}^d \to \mathbb{R}^d$ be a time-dependent vector field satisfying:*

(1) $\frac{|b_t(x)|}{1+|x|} \in L^\infty([0,T] \times \mathbb{R}^d)$;

(2) *$b_t \in \mathrm{BV}_{loc}(\mathbb{R}^d)$ for a.e. $t \in [0,T]$, and for any $R > 0$, $|Db_t|(B(R)) \in L^1_{loc}(0,T)$ and*

$$\int_0^T \|\mathrm{div}(b_t)\|_{L^\infty(B(R))} \mathrm{d}t < +\infty.$$

Then the vector field b generates a unique regular Lagrangian flow $\{Y_t : 0 \le t \le T\}$. Moreover for any $t \in [0,T]$, Y_t has a measurable inverse map $Y_t^{-1} : \mathbb{R}^d \to \mathbb{R}^d$ and $(Y_t^{-1})_\# \mathcal{L}_d = \rho_t \, \mathcal{L}_d$ with

$$\rho_t(x) = \exp\bigg(\int_0^t \mathrm{div}(b_s)(Y_s(x))\,\mathrm{d}s \bigg).$$

Proof. The first part of this theorem was first proved in [1] for bounded vector field b, and then in [2] for the general case (see the remark at the end of Section 5), while the second part was proved in [23] for a BV vector field b independent of time (just let the diffusion coefficients σ be 0 in Theorem 2.6), but the proof for the time-dependent case is similar. □

Remark 2.6. For the density function $\tilde{\rho}_t := \frac{\mathrm{d}((Y_t)_{\#}\mathcal{L}_d)}{\mathrm{d}\mathcal{L}_d}$, we have

$$\tilde{\rho}_t(x) = \exp\bigg(-\int_0^t \mathrm{div}(b_s)\big[Y_s(Y_t^{-1}(x))\big]\,\mathrm{d}s\bigg).$$

See [6] Theorem 2.1 where the expressions are given using double time parameters.

3 Existence and uniqueness of (1.1) with BV drift

In this section we prove Theorem 1.1. First we introduce Ocone and Pardoux's decomposition of the Stratonovich SDE (1.1) (see [21] PART II or [13] Section 2). Consider the following Stratonovich SDE without drift:

$$\mathrm{d}\tilde{X}_t = \sum_{i=1}^m A_i(\tilde{X}_t)\circ \mathrm{d}w_t^i, \quad \tilde{X}_0 = x.$$

It is well-known that under the conditions that $A_1, \cdots, A_m \in C_b^{3+\delta}$ for some $\delta > 0$, the solutions of the above SDE admit a version $\tilde{X}_t(x,w)$ such that there exists a full subset Ω_0, for each $w \in \Omega_0$ and each $t > 0$, $x \to \tilde{X}_t(x,w)$ is a C^2-diffeomorphism of $\mathbb{R}^d$. Set $\varphi_t(x) = \tilde{X}_t(x,w)$. Let $J_t(x) = \partial_x\varphi_t(x)$ be the Jacobian matrix of $\varphi_t : \mathbb{R}^d \to \mathbb{R}^d$ and $K_t(x) = (J_t(x))^{-1}$ the inverse of $J_t(x)$. Define for $w \in \Omega_0$,

$$\tilde{A}_0(t,x) = K_t(x)A_0(\varphi_t(x)). \tag{3.1}$$

We consider the differential equation

$$\mathrm{d}Y_t = \tilde{A}_0(t, Y_t)\,\mathrm{d}t, \quad Y_0 = x. \tag{3.2}$$

Then the solutions of (1.1) can be expressed as (at least when A_0 is smooth)

$$X_t(x) = \varphi_t(Y_t(x)). \tag{3.3}$$

Therefore it is sufficient to study the well posedness of the random ODE (3.2) under the assumptions on A_0 in Theorem 1.1, and then show that the representation (3.3) indeed gives the flow associated to the original Stratonovich SDE (1.1).

In the next lemma, we collect various growth results concerning the stochastic flow φ_t and its derivatives, and the random vector field $\tilde{A}_0$ defined in (3.1) (see [13] Lemma 2.2 for a proof).

Lemma 3.1. *Assume the conditions of Theorem 1.1, we have for any $T > 0$,*

(1) *for any $\alpha > 1$ and $\beta > 0$, there is $F, G \in \cap_{p>1} L^p(\Omega)$ such that for all $(t,x) \in [0,T]\times\mathbb{R}^d$,*

$$|\varphi_t(x)| \le F\cdot(1+|x|^\alpha), \quad \|J_t(x)\| \vee \|K_t(x)\| \le G\cdot(1+|x|^\beta);$$

(2) *there exist $\varepsilon_1 \in (0,1)$ and $\Phi_T \in \cap_{p>1} L^p(\Omega)$, such that for all $(t,x) \in [0,T]\times\mathbb{R}^d$,*

$$|\tilde{A}_0(t,x)| \le \Phi_T(1+|x|^{1-\varepsilon_1}).$$

Now we can prove

Proposition 3.2. *Under the conditions of Theorem 1.1, for almost surely $w \in \Omega_0$, the ODE* (3.2) *generates a unique regular Lagrangian flow Y_t which leaves the Lebesgue measure quasi-invariant; moreover for all $t \ge 0$, Y_t has a measurable inverse map $Y_t^{-1} : \mathbb{R}^d \to \mathbb{R}^d$.*

Proof. We only need to check that the conditions in Theorem 2.5 are satisfied for almost surely $w \in \Omega_0$. First by (2) of Lemma 3.1, the vector field $\tilde{A}_0$ satisfies the condition (1) in Theorem 2.5.

For all $t \in [0,T]$, since φ_t is a C^2-diffeomorphism on $\mathbb{R}^d$ and $K_t : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is C^1, Lemma 2.2 and the definition (3.1) of $\tilde{A}_0$ tell us that $\tilde{A}_0(t,\cdot) \in \mathrm{BV}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Moreover by the chain rule and Lemma 2.2 (1),

$$\begin{aligned} D\tilde{A}_0(t) &= (DK_t)A_0(\varphi_t) + K_t D(A_0(\varphi_t)) \\ &= (DK_t)A_0(\varphi_t) + |\det(J_t)|^{-1} K_t \,[(\varphi_t^{-1})_{\#} DA_0] J_t. \end{aligned} \tag{3.4}$$

Fix any $R > 0$. Recall that we identify a locally integrable function f on $\mathbb{R}^d$ with the Radon measure $f\,\mathrm{d}x$. Since $K_t \in C^1$, we know that $(t,x) \to DK_t(x)$ is bounded on $[0,T] \times B(R)$. Moreover, by the sublinear growth of A_0 and Lemma 3.1 (1), it is easy to deduce the boundedness of $A_0(\varphi_t)$ on $[0,T] \times B(R)$. Hence there exists a positive constant $C_{T,R}$ (depends on $w \in \Omega_0$) such that the total variation

$$|(DK_t)A_0(\varphi_t)|(B(R)) \le C_{T,R}\mathcal{L}_d(B(R)),$$

which implies that $t \to |(DK_t)A_0(\varphi_t)|(B(R)) \in L^1([0,T])$. Now we consider the second term on the right hand side of (3.4). Again the quantities $|\det(J_t)|^{-1}$, K_t and J_t^* are bounded on $[0,T] \times B(R)$, thus we only need to show that $t \to |(\varphi_t^{-1})_{\#} DA_0|(B(R))$ is integrable on $[0,T]$. We have

$$|(\varphi_t^{-1})_{\#} DA_0|(B(R)) = |DA_0|(\varphi_t(B(R))).$$

By (1) of Lemma 3.1, the set $\cup_{0\le t\le T}\varphi_t(B(R)) \subset B(F(1+R^\alpha))$ is bounded, from this we conclude that $[0,T] \ni t \to |(\varphi_t^{-1})_{\#} DA_0|(B(R))$ is a bounded function, hence integrable.

Now we check the last condition in (2) of Theorem 2.5. By Lemma 2.2 (2), we have

$$|\mathrm{div}(\tilde{A}_0(t))| \le |\langle \mathrm{div}(K_t), A_0(\varphi_t)\rangle| + |\mathrm{div}(A_0) \circ \varphi_t|.$$

Since $\mathrm{div}(K_t)(x)$ and $A_0(\varphi_t(x))$ are bounded on the product $[0,T] \times B(R)$, we know that $t \to \|\langle \mathrm{div}(K_t), A_0(\varphi_t)\rangle\|_{L^\infty(B(R))}$ is a integrable function on $[0,T]$. By the local boundedness of $\mathrm{div}(A_0)$ and Lemma 3.1 (1), we obtain the integrability of $[0,T] \ni t \to \|\mathrm{div}(A_0) \circ \varphi_t\|_{L^\infty(B(R))}$. Summing up these discussions, we arrive at

$$\int_0^T \|\mathrm{div}(\tilde{A}_0(t))\|_{L^\infty(B(R))} \mathrm{d}t < +\infty.$$

Therefore all the conditions of Theorem 2.5 are verified, and we complete the proof. □

Now we are at the position to give the

Proof of Theorem 1.1. (Existence) We only need to show that the flow $X_t(x) = \varphi_t(Y_t(x))$ satisfies the Stratonovich SDE (1.1). Remark that for all $w \in \Omega_0$ and any $t \in [0,T]$, the map $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is well defined almost everywhere. By the generalized Itô formula (see Theorem 3.3.2 in [17]) and the definitions of φ_t and Y_t, we have for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} \mathrm{d}X_t(x) &= (\mathrm{d}\varphi_t)(Y_t(x)) + (\partial_x\varphi_t)(Y_t(x))\,\mathrm{d}Y_t(x) \\ &= \left(\sum_{i=1}^m A_i(\varphi_t) \circ \mathrm{d}w_t^i\right)(Y_t(x)) + J_t(Y_t(x))K_t(Y_t(x))A_0\big[\varphi_t(Y_t(x))\big]\,\mathrm{d}t \\ &= \sum_{i=1}^m A_i(X_t(x)) \circ \mathrm{d}w_t^i + A_0(X_t(x))\,\mathrm{d}t. \end{aligned}$$

To show the quasi-invariance of the flow X_t, let $\tilde\rho_t$ be the Radon-Nikodym density of $(Y_t)_\#\mathcal{L}_d$ with respect to $\mathcal{L}_d$, then for any $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(X_t(x))\,\mathrm{d}x &= \int_{\mathbb{R}^d} \phi\big[\varphi_t(Y_t(x))\big]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \phi[\varphi_t(y)]\tilde\rho_t(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \phi(x)\big(\tilde\rho_t|\det(K_t)|\big)(\varphi_t^{-1}(x))\,\mathrm{d}x,\end{aligned}$$

where the last equality follows from the change of variable. Hence

$$(X_t)_\#\mathcal{L}_d = \big(\tilde\rho_t|\det(K_t)|\big)(\varphi_t^{-1})\,\mathcal{L}_d.$$

(Uniqueness) Suppose there is another solution Z_t, we consider $\tilde Z_t = \varphi_t^{-1}(Z_t)$. We will show that $\tilde Z_t$ solves the ODE (3.2). In fact, by [5] (see pp. 103–106, or (5.1) in [13]),

$$\mathrm{d}\varphi_t^{-1}(x) = -K_t\big(\varphi_t^{-1}(x)\big)\bigg(\sum_{i=1}^m A_i(x)\circ \mathrm{d}w_t^i\bigg). \tag{3.5}$$

Again by the generalized Itô formula ([17] Theorem 3.3.2),

$$\mathrm{d}\tilde Z_t = (\mathrm{d}\varphi_t^{-1})(Z_t) + \big[(\partial_x\varphi_t^{-1})(Z_t)\big]\circ \mathrm{d}Z_t.$$

Recall that for a.e. $x\in\mathbb{R}^d$, $Z_t(x)$ solves the Stratonovich SDE (1.1), and $\partial_x\varphi_t^{-1} = K_t\big(\varphi_t^{-1}(x)\big)$. Combining these results with (3.5) gives rise to

$$\begin{aligned}\mathrm{d}\tilde Z_t(x) = &-K_t\big(\tilde Z_t(x)\big)\bigg(\sum_{i=1}^m A_i(Z_t)\circ \mathrm{d}w_t^i\bigg) \\ &+K_t\big(\varphi_t^{-1}(Z_t(x))\big)\circ\bigg(\sum_{i=1}^m A_i(Z_t(x))\circ \mathrm{d}w_t^i + A_0(Z_t(x))\,\mathrm{d}t\bigg) \\ = &\,K_t(\tilde Z_t(x))A_0\big[\varphi_t(\tilde Z_t(x))\big]\mathrm{d}t = \tilde A_0(t,\tilde Z_t(x))\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

That is, $\tilde Z_t(x)$ solves the ODE (3.2) for a.e. $x\in\mathbb{R}^d$. But by Proposition 3.2, this equation generates a unique flow $Y_t:\mathbb{R}^d\to\mathbb{R}^d$. Hence $Y_t = \tilde Z_t = \varphi_t^{-1}(Z_t)$ for almost every $x\in\mathbb{R}^d$, which implies that any solution Z_t to (1.1) can be expressed as the composition $\varphi_t(Y_t)$. We get the uniqueness of (1.1). □

By Proposition 3.2, we know that the stochastic flow X_t in Theorem 1.1 has a inverse flow $X_t^{-1} = Y_t^{-1}\circ\varphi_t^{-1}$ which consists of measurable maps on $\mathbb{R}^d$.

4 Stability of (1.1) and stochastic transport equation

In this section we study the stability of the Stratonovich SDE, proving Theorem 1.2. As an application we will give an explicit solution to the corresponding stochastic transport equation.

Proposition 4.1. *Suppose the conditions of Theorem 1.2. For each $n\geq 1$, define $\tilde A_0^n(t,x) := K_t(x)A_0^n(\varphi_t(x))$. Let Y_t^n be the unique flow generated by the ODE (3.2) with $\tilde A_0$ being replaced by $\tilde A_0^n$. Then almost surely, for any T, $R>0$, we have*

$$\lim_{n\to\infty}\int_{B(R)}\sup_{0\leq t\leq T}|Y_t^n(x)-Y_t(x)|\,\mathrm{d}x = 0.$$

Proof. As in Proposition 3.2, now we check the conditions in [1] Theorem 6.6. It is clear that condition (6.3) is verified. Remark again that the uniform boundedness assumption in (6.4) can be relaxed to allow uniform linear growth. Since the vector fields A_0^n have the same sublinear growth, we can prove a uniform growth estimate for $\tilde{A}_0^n(t,x)$ similar to the one given in (2) of Lemma 3.1 (see [13] Lemma 2.2 for a proof). Next we check that $\tilde{A}_0^n$ converges in $L^1_{loc}((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ to $\tilde{A}_0$ defined in (3.1). We have

$$|\tilde{A}_0^n(t,x)-\tilde{A}_0(t,x)|\leq\|K_t(x)\|\cdot|A_0^n(\varphi_t(x))-A_0(\varphi_t(x))|.$$

Since $(t,x)\to K_t(x)$ is continuous on $[0,T]\times B(R)$, there is $C_{T,R}>0$ such that

$$\sup\{\|K_t(x)\|:(t,x)\in[0,T]\times B(R)\}\leq C_{T,R}.$$

Therefore

$$\begin{aligned}\int_0^T\!\!\int_{B(R)}|\tilde{A}_0^n(t,x)-\tilde{A}_0(t,x)|\,\mathrm{d}x\mathrm{d}t &\leq C_{T,R}\int_0^T\!\!\int_{B(R)}|A_0^n(\varphi_t(x))-A_0(\varphi_t(x))|\,\mathrm{d}x\mathrm{d}t\\ &= C_{T,R}\int_0^T\!\!\int_{\varphi_t(B(R))}|A_0^n-A_0|\cdot|\det(K_t(\varphi_t^{-1}))|\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By Lemma 3.1, the set $\cup_{0\leq t\leq T}\varphi_t(B(R))\subset B\big(F(1+R^\alpha)\big)$ is bounded, and the function $|\det(K_t(x))|$ is bounded on $[0,T]\times B(R)$. As a consequence,

$$\int_0^T\!\!\int_{B(R)}|\tilde{A}_0^n(t,x)-\tilde{A}_0(t,x)|\,\mathrm{d}x\leq C'_{T,R}T\int_{B(F(1+R^\alpha))}|A_0^n-A_0|\,\mathrm{d}x\mathrm{d}t,$$

which, by condition (2) of Theorem 1.2, converges to 0 for almost surely $w\in\Omega_0$.

Now we verify the condition (6.5) of Theorem 6.6 in [1]. By the definition of $\tilde{A}_0^n(t,x)$, it holds

$$\nabla\tilde{A}_0^n(t,x)=(\nabla K_t(x))A_0^n(\varphi_t(x))+K_t(x)(\nabla A_0^n)(\varphi_t(x))J_t(x),$$

hence

$$\|\nabla\tilde{A}_0^n(t,x)\|\leq\|\nabla K_t(x)\|\cdot|A_0^n(\varphi_t(x))|+\|K_t(x)\|\cdot\|J_t(x)\|\cdot\|(\nabla A_0^n)(\varphi_t(x))\|.$$

The terms $\|\nabla K_t(x)\|$, $\|K_t(x)\|$ and $\|J_t(x)\|$ are bounded on $[0,T]\times B(R)$. By Lemma 3.1 and the fact that A_0^n have uniform growth, it is easy to show that the sequence $|A_0^n(\varphi_t(x))|$ has an upper bound on $[0,T]\times B(R)$ independent of n. Regarding the last term, notice again that $\cup_{0\leq t\leq T}\varphi_t(B(R))$ is a bounded subset and that $\|\nabla A_0^n\|$ are locally bounded. Summing up the above arguments we obtain the boundedness of $\nabla\tilde{A}_0^n(t,x)$ on $[0,T]\times B(R)$.

Finally to verify the condition (6.6) in [1], in view of Remark 6.3, it is sufficient to show that for all $R>0$

$$C_T:=\sup_{n\geq1}\int_0^T\|\mathrm{div}(\tilde{A}_0^n(t,\cdot))\|_{L^\infty(B(R))}\,\mathrm{d}t<+\infty.$$

By the definition of $\tilde{A}_0^n$ and (2) of Lemma 2.2, we have

$$|\mathrm{div}(\tilde{A}_0^n(t,\cdot))|\leq|\mathrm{div}(K_t)|\cdot|A_0^n\circ\varphi_t|+|\mathrm{div}(A_0^n)\circ\varphi_t|.$$

Similar discussions as above lead to the desired result. Thus all the conditions in [1] Theorem 6.6 are satisfied, and we complete the proof. □

Corollary 4.2. *Under the conditions of Theorem 1.2, for any $p \geq 1$, almost surely*

$$\lim_{n\to\infty} \int_{B(R)} \sup_{0\leq t\leq T} |Y_t^n(x) - Y_t(x)|^p \, \mathrm{d}x = 0.$$

Proof. By Proposition 4.1, the sequence $\sup_{0\leq t\leq T} |Y_t^n(\cdot) - Y_t(\cdot)|$ converges to 0 in the Lebesgue measure on the ball $B(R)$. Hence we only need to show that this sequence is bounded in $L^p(B(R), \mathrm{d}x)$ for any $p > 1$, then the desired result follows from the uniform integrability.

By the growth estimate of $\tilde{A}_0$ in Lemma 3.1, it is easy to deduce that $|Y_t(x)| \leq \tilde{\Phi} \cdot (1 + |x|)$, where $\tilde{\Phi} \in \cap_{p>1} L^p(\Omega)$ (see (iii) in the proof of [19] Lemma 3.2). Remark that for every $n \geq 1$, $\tilde{A}_0^n$ has the same growth as $\tilde{A}_0$, hence $\sup_{n\geq 1} |Y_t^n(x)| \leq \tilde{\Phi} \cdot (1 + |x|)$. Therefore

$$\sup_{0\leq t\leq T} |Y_t^n(x) - Y_t(x)|^p \leq 2^p \tilde{\Phi}^p (1 + |x|^p), \tag{4.1}$$

which implies clearly the boundedness of the sequence in $L^p(B(R), \mathrm{d}x)$. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. We use the representations of the solutions: $X_t = \varphi_t(Y_t)$ and $X_t^n = \varphi_t(Y_t^n)$. By the mean value formula,

$$\begin{aligned} X_t^n(x) - X_t(x) &= \varphi_t(Y_t^n(x)) - \varphi_t(Y_t(x)) \\ &= \left(\int_0^1 J_t\big((1-u)Y_t(x) + uY_t^n(x)\big) \mathrm{d}u \right) (Y_t^n(x) - Y_t(x)). \end{aligned} \tag{4.2}$$

By Lemma 3.1 and the growth estimates of Y_t and Y_t^n given in the proof of Corollary 4.2,

$$\begin{aligned} \big\| J_t\big((1-u)Y_t(x) + uY_t^n(x)\big) \big\| &\leq G\big(1 + |Y_t(x)|^\beta + |Y_t^n(x)|^\beta\big) \\ &\leq G\big(1 + 2\tilde{\Phi}^\beta\big)\big(1 + |x|^\beta\big). \end{aligned}$$

Therefore by (4.2),

$$\begin{aligned} |X_t^n(x) - X_t(x)| &\leq \left(\int_0^1 \big\| J_t\big((1-u)Y_t(x) + uY_t^n(x)\big) \big\| \mathrm{d}u \right) |Y_t^n(x) - Y_t(x)| \\ &\leq G\big(1 + 2\tilde{\Phi}^\beta\big)\big(1 + |x|^\beta\big) |Y_t^n(x) - Y_t(x)|. \end{aligned} \tag{4.3}$$

As a result, for $w \in \Omega_0$,

$$\int_{B(R)} \sup_{0\leq t\leq T} |X_t^n(x) - X_t(x)| \, \mathrm{d}x \leq G\big(1 + 2\tilde{\Phi}^\beta\big)\big(1 + |R|^\beta\big) \int_{B(R)} \sup_{0\leq t\leq T} |Y_t^n(x) - Y_t(x)| \, \mathrm{d}x,$$

which converges to 0 almost surely by Proposition 4.1.

Now we prove the $L^p(\Omega)$ convergence of solutions for any $p \geq 1$. Indeed we will prove a stronger result: for any $p \geq 1$,

$$\lim_{n\to\infty} \mathbb{E} \int_{B(R)} \sup_{0\leq t\leq T} |X_t^n(x) - X_t(x)|^p \, \mathrm{d}x = 0.$$

Similar to Corollary 4.2, it is enough to show that the sequence $\big\{ \int_{B(R)} \sup_{0\leq t\leq T} |X_t^n(x) - X_t(x)|^p \, \mathrm{d}x : n \geq 1 \big\}$ is bounded in some $L^q(\Omega)$ $(q > 1)$. However this follows easily from (4.1), (4.3) and the facts that G, $\tilde{\Phi}$ belong to all $L^q(\Omega)$. The proof of Theorem 1.2 is complete. □

As an application of the stability result, now we study the corresponding stochastic transport equation with the purpose of constructing an explicit solution to it, by using the flow generated

by (1.1). First we give some remarks on the "inverse" flow associated to (1.1). To this end we regularize the drift vector field A_0 by convolution using a standard kernel χ_n: $A_0^n = \phi_n\cdot(A_0*\chi_n)$, here $\phi_n(x) = \phi(x/n)$ with $\phi \in C_c^\infty(\mathbb{R}^d, [0,1])$ satisfying

$$\phi|_{B(1)} \equiv 1, \quad \mathrm{supp}(\phi) \subset B(2).$$

Let $X_t^n(x,w)$ be the smooth flow associated to (1.1) with A_0 being replaced by A_0^n. Then it is clear that the conditions in Theorem 1.2 are satisfied, hence for any $p > 1$ and $T, R > 0$, we have

$$\lim_{n\to\infty} \mathbb{E}\int_{B(R)} \sup_{0\le t\le T} |X_t^n(x) - X_t(x)|^p \,\mathrm{d}x = 0. \tag{4.4}$$

Fix some $T > 0$, define the time-reversed Brownian motion $\hat w_t^T = w_T - w_{T-t}$ and consider

$$\mathrm{d}\hat X_t^T = \sum_{i=1}^m A_i\big(\hat X_t^T\big)\circ \mathrm{d}\big(\hat w_t^T\big)^i - A_0\big(\hat X_t^T\big)\,\mathrm{d}t, \quad X_0 = x. \tag{4.5}$$

Under the conditions of Theorem 1.1, this equation still generates a unique flow $\hat X_t^T$, $0 \le t \le T$. Similarly we have $\hat X_t^{n,T}\big(x, \hat w^T\big)$ which is the solution of the above equation by replacing A_0 with A_0^n, then we have $(X_T^n)^{-1} = \hat X_T^{n,T}$ and

$$\lim_{n\to\infty} \mathbb{E}\int_{B(R)} \sup_{0\le t\le T} |\hat X_t^{n,T}(x) - \hat X_t^T(x)|^p \,\mathrm{d}x = 0. \tag{4.6}$$

With the convergence results (4.4) and (4.6) in hand, we can prove

Lemma 4.3. *Assume the conditions of Theorem 1.1 and that the divergence* $\mathrm{div}(A_0)$ *is bounded on* $\mathbb{R}^d$. *Then for every* $T > 0$, $X_T^{-1} = \hat X_T^T$ *a.e. on* $\mathbb{R}^d$, *and the density function* σ_T *of* $(X_T^{-1})_\#\mathcal{L}_d$ *with respect to* $\mathcal{L}_d$ *has the expression:*

$$\sigma_T(x) = \exp\bigg(\sum_{i=1}^m \int_0^T \mathrm{div}(A_i)(X_s(x))\circ \mathrm{d}w_s^i + \int_0^T \mathrm{div}(A_0)(X_s(x))\,\mathrm{d}s\bigg).$$

Proof. For every $n \ge 1$, we have $((X_t^n)^{-1})_\#\mathcal{L}_d = \sigma_t^{(n)}\mathcal{L}_d$ where (see Lemma 4.3.1 in [17])

$$\sigma_t^{(n)}(x) = \exp\bigg(\sum_{i=1}^m \int_0^t \mathrm{div}(A_i)(X_s^n(x))\circ \mathrm{d}w_s^i + \int_0^t \mathrm{div}(A_0^n)(X_s^n(x))\,\mathrm{d}s\bigg).$$

Next $\mathrm{div}(A_0^n) = \langle\nabla\phi_n, A_0*\chi_n\rangle + \phi_n\cdot(\mathrm{div}(A_0)*\chi_n)$. Since A_0 has sublinear growth, it is obvious that there exists $C > 0$ such that $\sup_{n\ge1}|(A_0*\chi_n)(x)| \le C(1+|x|)$. By the definition of ϕ_n,

$$\begin{aligned}|\langle\nabla\phi_n, A_0*\chi_n\rangle| &\le \frac{1}{n}|\nabla\phi(\cdot/n)|\cdot|A_0*\chi_n| \\ &\le \frac{C\|\nabla\phi\|_\infty}{n}(1+|x|)\mathbf{1}_{\{n\le|x|\le2n\}} \le 3C\|\nabla\phi\|_\infty.\end{aligned}$$

Hence the divergences

$$|\mathrm{div}(A_0^n)| \le |\langle\nabla\phi_n, A_0*\chi_n\rangle| + |\phi_n\cdot(\mathrm{div}(A_0)*\chi_n)| \le 3C\|\nabla\phi\|_\infty + \|\mathrm{div}(A_0)\|_\infty$$

are uniformly bounded on $\mathbb{R}^d$. As a result (see Lemma 3.5 in [19]), for any $p \in \mathbb{R}$,

$$\sup_{n\ge1}\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathbb{E}\big[(\sigma_t^{(n)}(x))^p\big] < +\infty.$$

Now similar arguments as in the proof of Theorem 2.6 in [23] lead to the result. □

This lemma tells us that if $\mathrm{div}(A_0)$ is bounded, then the flow we constructed in Theorem 1.1 is indeed an almost everywhere stochastic invertible flow in the sense of Definition 2.1 in [23]. Furthermore by (4.6), for any $t > 0$, it holds

$$\lim_{n\to\infty} \mathbb{E} \int_{B(R)} \big|(X_t^n)^{-1}(x) - X_t^{-1}(x)\big|^p \,\mathrm{d}x = 0. \tag{4.7}$$

Now we can construct an explicit solution to the corresponding stochastic transport equation by using the inverse flow X_t^{-1}, as in Theorem 5.1 of [13]. Though we follow the idea of the proof of [13] Theorem 5.1, the difference is that here we only assume that the initial value θ_0 is measurable, hence the proof of (4.19) is much more delicate than (5.18) in [13] (see [23] Proposition 2.3 for a different method, but θ_0 is supposed to be bounded there).

Proposition 4.4. *Assume the conditions in Theorem 1.1 and that* $\mathrm{div}(A_0)$ *is bounded. Let* $\theta_0 : \mathbb{R}^d \to \mathbb{R}$ *be a measurable function with polynomial growth. Then* $\theta(t,x) := \theta_0(X_t^{-1}(x))$ *is a distributional solution to the following stochastic transport equation:*

$$\mathrm{d}\theta(t) = -\sum_{i=1}^m \langle \nabla\theta(t), A_i\rangle \circ \mathrm{d}w_t^i - \langle \nabla\theta(t), A_0\rangle \,\mathrm{d}t, \quad \theta|_{t=0} = \theta_0, \tag{4.8}$$

that is, for any $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$(\theta(t),\phi)_{L^2} = (\theta_0,\phi)_{L^2} + \sum_{i=1}^m \int_0^t (\theta(s), \mathrm{div}(\phi A_i))_{L^2} \circ \mathrm{d}w_s^i + \int_0^t (\theta(s), \mathrm{div}(\phi A_0))_{L^2} \mathrm{d}s. \tag{4.9}$$

where $(\cdot,\cdot)_{L^2}$ *is the inner product in* $L^2(\mathbb{R}^d, \mathrm{d}x)$.

Proof. First we transform the equation (4.9) into the Itô form:

$$\begin{aligned}(\theta(t),\phi)_{L^2} = {} & (\theta_0,\phi)_{L^2} + \sum_{i=1}^m \int_0^t (\theta(s), \mathrm{div}(\phi A_i))_{L^2} \mathrm{d}w_s^i + \int_0^t (\theta(s), \mathrm{div}(\phi A_0))_{L^2} \mathrm{d}s \\ & + \frac{1}{2} \sum_{i=1}^m \int_0^t \big(\theta(s), \mathrm{div}(\mathrm{div}(\phi A_i) A_i)\big)_{L^2} \mathrm{d}s. \end{aligned} \tag{4.10}$$

The proof is similar to that of Theorem 5.1 in [13] and we divide it into two parts.

Step 1. We assume $\theta_0 \in C_c^\infty$. In this step, similar to the proof of Theorem 5.1 in [13], the key point is to show

$$\lim_{n\to\infty} \mathbb{E}\bigg(\sup_{|x|\le R} |\theta_n(t,x) - \theta(t,x)|^p \bigg) = 0,$$

where $\theta_n(t,x) = \theta_0\big((X_t^n)^{-1}(x)\big)$ and X_t^n is defined before Lemma 4.3. However, similar to (5.12) on p.1102 of [13], we only need a weaker form of the convergence result:

$$\lim_{n\to\infty} \mathbb{E} \int_{B(R)} |\theta_n(t,x) - \theta(t,x)|^p \,\mathrm{d}x = 0. \tag{4.11}$$

To this end, notice that $|\theta_n(t,x) - \theta(t,x)| \le \|\nabla\theta_0\|_\infty |(X_t^n)^{-1}(x) - X_t^{-1}(x)|$, thus by (4.7), we still have

$$\mathbb{E} \int_{B(R)} |\theta_n(t,x) - \theta(t,x)|^p \,\mathrm{d}x \le \|\nabla\theta_0\|_\infty^p \,\mathbb{E} \int_{B(R)} |(X_t^n)^{-1}(x) - X_t^{-1}(x)|^p \mathrm{d}x \to 0,$$

as $n \to \infty$. Hence (4.11) holds and the rest of the arguments in Step 1 of the proof of Theorem 5.1 in [13] still work here.

Step 2. Now suppose that θ_0 is measurable with polynomial growth. Define $\theta_0^n = \phi_n\,(\theta_0 * \chi_n)$. Then there exist $C > 0$ and $q_0 \in \mathbb{Z}_+$ which are independent of n such that

$$|\theta_0^n(x)| \le C\,(1 + |x|^{q_0}), \quad x \in \mathbb{R}^d. \tag{4.12}$$

Use again the notation $\theta_n(t,x)$ to denote $\theta_n(t,x) = \theta_0^n(X_t^{-1}(x))$ where $X_t(x)$ is now the solution to SDE (1.1). Then by Step 1, θ_n satisfies (4.10). Now using the SDE (4.5) and the moment estimate, we have for any $T > 0$ and $t \in [0,T]$,

$$\mathbb{E}\bigg(\sup_{0\le s\le t} |\hat{X}_s^t(x)|^p\bigg) \le C_{p,T}\,(1 + |x|^p).$$

In particular, for each $t \in [0,T]$,

$$\mathbb{E}\big(|X_t^{-1}(x)|^p\big) \le C_{p,T}\,(1 + |x|^p).$$

By (4.12), it holds that

$$\sup_{t\le T} \mathbb{E}\big(|\theta_n(t,x)|^p\big) + \sup_{t\le T} \mathbb{E}\big(|\theta(t,x)|^p\big) \le C_{p,T}\,(1 + |x|^{pq_0}). \tag{4.13}$$

Therefore for any $p > 2$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\phi) \subset B(R)$,

$$\begin{aligned}\int_0^T \mathbb{E}\big(|(\theta_n(t),\phi)_{L^2}|^p\big)\,\mathrm{d}t &\le \bigg(\int_{\mathbb{R}^d} |\phi|^q\,\mathrm{d}x\bigg)^{p-1} \int_0^T\!\!\int_{B(R)} \mathbb{E}(|\theta_n(t,x)|^p)\,\mathrm{d}x\mathrm{d}t \\ &\le C_{p,T}\bigg(\int_{\mathbb{R}^d} |\phi|^q\,\mathrm{d}x\bigg)^{p-1} \int_{B(R)} (1 + |x|^{pq_0})\,\mathrm{d}x < +\infty. \end{aligned} \tag{4.14}$$

where q is the conjugate number of p.

Fix some $M > 0$. We have

$$\begin{aligned}\mathbb{E}\int_{B(R)} |\theta_n(t,x) - \theta(t,x)|^2\mathrm{d}x &= \mathbb{E}\int_{X_t^{-1}(B(R))} |\theta_0^n(y) - \theta_0(y)|^2\sigma_t(y)\,\mathrm{d}y \\ &\le \mathbb{E}\bigg(\int_{B(M)} + \int_{X_t^{-1}(B(R))\setminus B(M)}\bigg)|\theta_0^n(y) - \theta_0(y)|^2\sigma_t(y)\,\mathrm{d}y \\ &=: I_1^n(t) + I_2^n(t). \end{aligned} \tag{4.15}$$

Since $\mathrm{div}(A_0)$ is bounded, we have by Lemma 3.5 in [19] that

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} \mathbb{E}(\sigma_t(x)) \le C_T < +\infty,$$

hence

$$\begin{aligned}\limsup_{n\to\infty} I_1^n(t) &= \limsup_{n\to\infty} \int_{B(M)} |\theta_0^n(y) - \theta_0(y)|^2\mathbb{E}(\sigma_t(y))\,\mathrm{d}y \\ &\le C_T \lim_{n\to\infty} \int_{B(M)} |\theta_0^n(y) - \theta_0(y)|^2\,\mathrm{d}y = 0, \end{aligned} \tag{4.16}$$

where the last equality follows from the convergence of θ_0^n to θ_0 in $L^2_{loc}(\mathbb{R}^d)$. By the Cauchy inequality,

$$I_2^n(t) \le \bigg(\mathbb{E}\int_{X_t^{-1}(B(R))\setminus B(M)} |\theta_0^n(y) - \theta_0(y)|^4\sigma_t(y)\,\mathrm{d}y\bigg)^{1/2}\bigg(\mathbb{E}\int_{X_t^{-1}(B(R))\setminus B(M)} \sigma_t(y)\,\mathrm{d}y\bigg)^{1/2}$$

$$=: \big(I^n_{2,1}(t) I^n_{2,2}(t)\big)^{1/2}. \tag{4.17}$$

We have by (4.13),

$$\begin{aligned} I^n_{2,1}(t) &\leq \mathbb{E}\int_{X_t^{-1}(B(R))} |\theta^n_0(y)-\theta_0(y)|^4 \sigma_t(y)\,\mathrm{d}y \\ &= \mathbb{E}\int_{B(R)} \big|\theta^n_0(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))\big|^4\,\mathrm{d}x \\ &\leq C' \int_{B(R)} (1+|x|^{4q_0})\,\mathrm{d}x < +\infty. \end{aligned} \tag{4.18}$$

Next the function $\sigma_t \mathbf{1}_{X_t^{-1}(B(R))\setminus B(M)}$ tends to 0 as M tends to $+\infty$ for $\mathbb{P}\times\mathcal{L}_d$-a.e. $(w,y)\in\Omega_0\times\mathbb{R}^d$; moreover $\sigma_t \mathbf{1}_{X_t^{-1}(B(R))\setminus B(M)} \leq \sigma_t \mathbf{1}_{X_t^{-1}(B(R))}$ and

$$\mathbb{E}\int_{\mathbb{R}^d} \sigma_t \mathbf{1}_{X_t^{-1}(B(R))}\,\mathrm{d}y = \mathbb{E}\int_{\mathbb{R}^d} \mathbf{1}_{B(R)}\mathrm{d}y = \mathcal{L}_d(B(R)) < +\infty.$$

Hence by the dominated convergence theorem, we have

$$\lim_{M\to+\infty} I^n_{2,2}(t) = 0.$$

This plus (4.17) and (4.18) tells us that

$$\lim_{M\to+\infty} I^n_2(t) = 0.$$

Therefore by (4.16), first letting n goes to $+\infty$ in (4.15), and then letting M goes to ∞, we obtain

$$\lim_{n\to+\infty} \mathbb{E}\int_{B(R)} |\theta_n(t,x)-\theta(t,x)|^2\mathrm{d}x = 0.$$

From this we deduce that for any $\phi\in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\phi)\subset B(R)$,

$$\mathbb{E}\big[\big((\theta_n(t),\phi)_{L^2} - (\theta(t),\phi)_{L^2}\big)^2\big] \leq \bigg(\int_{\mathbb{R}^d}\phi^2\mathrm{d}x\bigg)\mathbb{E}\int_{B(R)} |\theta_n(t,x)-\theta(t,x)|^2\mathrm{d}x \to 0 \tag{4.19}$$

as $n\to\infty$. Now (4.14) and (4.19) allow us to pass to the limit and the proof is complete. $\square$

Now we discuss the connection between the stochastic transport equation (4.8) and the following transport equation associated to the random vector field $\tilde{A}_0$ defined in (3.1):

$$\mathrm{d}u_t = -\langle \nabla u_t, \tilde{A}_0(t)\rangle\,\mathrm{d}t, \quad u|_{t=0} = u_0. \tag{4.20}$$

To this end we first give some preparations. Recall that φ_t is the smooth flow defined at the beginning of Section 3. Let $\tilde\rho_t = \frac{\mathrm{d}((\varphi_t)_\#\mathcal{L}_d)}{\mathrm{d}\mathcal{L}_d}$ and $\rho_t = \frac{\mathrm{d}((\varphi_t^{-1})_\#\mathcal{L}_d)}{\mathrm{d}\mathcal{L}_d}$ be the Radon-Nikodym densities. We have the following simple equality:

$$\tilde\rho_t(x) = \big[\rho_t\big(\varphi_t^{-1}(x)\big)\big]^{-1}. \tag{4.21}$$

Indeed, for any $\psi\in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \psi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \psi\big[\varphi_t\big(\varphi_t^{-1}(x)\big)\big]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \psi[\varphi_t(y)]\rho_t(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \psi(x)\rho_t\big(\varphi_t^{-1}(x)\big)\tilde\rho_t(x)\,\mathrm{d}x, \end{aligned}$$

which leads to (4.21) due to the arbitrariness of $\psi \in C_c^\infty(\mathbb{R}^d)$. Moreover by Lemma 4.3.1 in [17], the density ρ_t has an explicit expression:

$$\rho_t(x) = \exp\bigg(\sum_{i=1}^m \int_0^t \mathrm{div}(A_i)(\varphi_s(x)) \circ \mathrm{d}w_s^i\bigg). \tag{4.22}$$

Now we show that the distributional solutions of (4.8) and (4.20) are related to each other by the smooth flow φ_t.

Proposition 4.5. *Suppose that θ_t is a distributional solution to the stochastic transport equation* (4.8)*, then almost surely, $u_t := \theta_t(\varphi_t)$ solves the transport equation* (4.20) *with $u|_{t=0} = \theta_0$.*

Proof. For any $\psi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \theta_t(\varphi_t)\psi \,\mathrm{d}x = \int_{\mathbb{R}^d} \theta_t \psi(\varphi_t^{-1})\tilde\rho_t \,\mathrm{d}x. \tag{4.23}$$

Now we compute the Stratonovich stochastic differentials of $\psi(\varphi_t^{-1})$ and $\tilde\rho_t$. By [5] (see pp. 103–106, or (5.1) in [13]),

$$\mathrm{d}\varphi_t^{-1} = -K_t(\varphi_t^{-1}) \sum_{i=1}^m A_i(x) \circ \mathrm{d}w_t^i. \tag{4.24}$$

Hence

$$\mathrm{d}\psi(\varphi_t^{-1}) = \big\langle (\nabla\psi)(\varphi_t^{-1}), \circ\, \mathrm{d}\varphi_t^{-1}\big\rangle = -\sum_{i=1}^m \big\langle (\nabla\psi)(\varphi_t^{-1}), K_t(\varphi_t^{-1})A_i\big\rangle \circ \mathrm{d}w_t^i.$$

Notice that $\nabla(\psi(\varphi_t^{-1})) = K_t^*(\varphi_t^{-1})(\nabla\psi)(\varphi_t^{-1})$, we obtain

$$\mathrm{d}\psi(\varphi_t^{-1}) = -\sum_{i=1}^m \big\langle \nabla(\psi(\varphi_t^{-1})), A_i\big\rangle \circ \mathrm{d}w_t^i. \tag{4.25}$$

Next we compute $\mathrm{d}\tilde\rho_t$. By (4.22),

$$\mathrm{d}\rho_t = \rho_t \sum_{i=1}^m \mathrm{div}(A_i)(\varphi_t) \circ \mathrm{d}w_t^i,$$

hence we deduce from (4.24) and the generalized Itô formula that

$$\begin{aligned} \mathrm{d}\big[\rho_t(\varphi_t^{-1})\big] &= (\mathrm{d}\rho_t)(\varphi_t^{-1}) + \big\langle (\nabla\rho_t)(\varphi_t^{-1}), \circ\, \mathrm{d}\varphi_t^{-1}\big\rangle \\ &= \rho_t(\varphi_t^{-1}) \sum_{i=1}^m \mathrm{div}(A_i) \circ \mathrm{d}w_t^i - \sum_{i=1}^m \big\langle (K_t^*\nabla\rho_t)(\varphi_t^{-1}), A_i\big\rangle \circ \mathrm{d}w_t^i. \end{aligned}$$

Using again the Itô formula and by the relation (4.21), we arrive at

$$\begin{aligned} \mathrm{d}\tilde\rho_t &= -\big[\rho_t(\varphi_t^{-1})\big]^{-2} \circ \mathrm{d}\big[\rho_t(\varphi_t^{-1})\big] \\ &= -\tilde\rho_t \sum_{i=1}^m \mathrm{div}(A_i) \circ \mathrm{d}w_t^i + \tilde\rho_t^2 \sum_{i=1}^m \big\langle \nabla(\rho_t(\varphi_t^{-1})), A_i\big\rangle \circ \mathrm{d}w_t^i. \end{aligned}$$

Since $\nabla\tilde\rho_t = -\tilde\rho_t^2 \nabla(\rho_t(\varphi_t^{-1}))$, finally we obtain

$$\mathrm{d}\tilde\rho_t = -\tilde\rho_t \sum_{i=1}^m \mathrm{div}(A_i) \circ \mathrm{d}w_t^i - \sum_{i=1}^m \langle \nabla\tilde\rho_t, A_i\rangle \circ \mathrm{d}w_t^i = -\sum_{i=1}^m \mathrm{div}(\tilde\rho_t A_i) \circ \mathrm{d}w_t^i. \tag{4.26}$$

Now by the equalities (4.25), (4.26) and the fact that θ_t solves the stochastic transport equation (4.8), we have

$$\begin{aligned}\mathrm{d}\big[\theta_t\psi(\varphi_t^{-1})\tilde\rho_t\big] &= \psi(\varphi_t^{-1})\tilde\rho_t\bigg(-\sum_{i=1}^m\langle\nabla\theta_t, A_i\rangle\circ \mathrm{d}w_t^i - \langle\nabla\theta_t, A_0\rangle\,\mathrm{d}t\bigg)\\ &\quad -\sum_{i=1}^m \theta_t\tilde\rho_t\langle\nabla(\psi(\varphi_t^{-1})), A_i\rangle\circ\mathrm{d}w_t^i - \sum_{i=1}^m\theta_t\psi(\varphi_t^{-1})\mathrm{div}(\tilde\rho_t A_i)\circ\mathrm{d}w_t^i.\end{aligned}$$

Since $\mathrm{div}(\psi(\varphi_t^{-1})\tilde\rho_t A_i) = \psi(\varphi_t^{-1})\mathrm{div}(\tilde\rho_t A_i) + \tilde\rho_t\langle\nabla(\psi(\varphi_t^{-1})), A_i\rangle$, we arrive at

$$\begin{aligned}\mathrm{d}\big[\theta_t\psi(\varphi_t^{-1})\tilde\rho_t\big] &= \psi(\varphi_t^{-1})\tilde\rho_t\bigg(-\sum_{i=1}^m\langle\nabla\theta_t, A_i\rangle\circ \mathrm{d}w_t^i - \langle\nabla\theta_t, A_0\rangle\,\mathrm{d}t\bigg)\\ &\quad -\sum_{i=1}^m \theta_t\mathrm{div}\big(\psi(\varphi_t^{-1})\tilde\rho_t A_i\big)\circ\mathrm{d}w_t^i.\end{aligned}$$

The above equality should be understood in the distributional sense. More precisely we have obtained

$$\begin{aligned}\int_{\mathbb{R}^d}\theta_t\psi(\varphi_t^{-1})\tilde\rho_t\,\mathrm{d}x &= \int_{\mathbb{R}^d}\theta_0\psi\,\mathrm{d}x + \sum_{i=1}^m\int_0^t\bigg(\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_i\big)\mathrm{d}x\bigg)\circ\mathrm{d}w_s^i\\ &\quad + \int_0^t\!\!\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_0\big)\mathrm{d}x\mathrm{d}s\\ &\quad -\sum_{i=1}^m\int_0^t\bigg(\int_{\mathbb{R}^d}\theta_s\mathrm{div}(\psi(\varphi_s^{-1})\tilde\rho_s A_i)\mathrm{d}x\bigg)\circ\mathrm{d}w_s^i\\ &= \int_{\mathbb{R}^d}\theta_0\psi\,\mathrm{d}x + \int_0^t\!\!\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_0\big)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

As in Proposition 4.4 we denote by $(\cdot,\cdot)_{L^2}$ the inner product in $L^2(\mathbb{R}^d, \mathrm{d}x)$. By (4.23) we have

$$(\theta_t(\varphi_t), \psi)_{L^2} = (\theta_0, \psi)_{L^2} + \int_0^t\!\!\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_0\big)\mathrm{d}x\mathrm{d}s. \tag{4.27}$$

By the definition of $\tilde\rho_s$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_0\big)\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^d}\theta_s\tilde\rho_s\big[\big\langle(K_s^*\nabla\psi)(\varphi_s^{-1}), A_0\big\rangle + \psi(\varphi_s^{-1})\tilde\rho_s^{-1}\langle\nabla\tilde\rho_s, A_0\rangle + \psi(\varphi_s^{-1})\mathrm{div}(A_0)\big]\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^d}\theta_s(\varphi_s)\big[\big\langle\nabla\psi, K_s A_0(\varphi_s)\big\rangle + \psi\big\langle(\tilde\rho_s^{-1}\nabla\tilde\rho_s)(\varphi_s), A_0(\varphi_s)\big\rangle + \psi\mathrm{div}(A_0)(\varphi_s)\big]\mathrm{d}x.\end{aligned}$$

Lemma 2.3 leads to

$$\begin{aligned}\mathrm{div}\big(\tilde A_0(s)\big) &= \langle\mathrm{div}(K_s), A_0(\varphi_s)\rangle + \mathrm{div}(A_0)(\varphi_s)\\ &= \big\langle(\tilde\rho_s^{-1}\nabla\tilde\rho_s)(\varphi_s), A_0(\varphi_s)\big\rangle + \mathrm{div}(A_0)(\varphi_s),\end{aligned}$$

it follows that

$$\int_{\mathbb{R}^d}\theta_s\mathrm{div}\big(\psi(\varphi_s^{-1})\tilde\rho_s A_0\big)\mathrm{d}x = \int_{\mathbb{R}^d}\theta_s(\varphi_s)\mathrm{div}\big(\psi\tilde A_0(s)\big)\mathrm{d}x.$$

This plus (4.27) gives us

$$(\theta_t(\varphi_t), \psi)_{L^2} = (\theta_0, \psi)_{L^2} + \int_0^t\big(\theta_s(\varphi_s), \mathrm{div}(\psi\tilde A_0(s))\big)_{L^2}\,\mathrm{d}s,$$

which means that almost surely, $\theta_t(\varphi_t)$ is a distributional solution to the transport equation (4.20) with initial value θ_0. □

Remark 4.6. Originally we intended to prove the uniqueness of the solutions to the stochastic transport equation (4.8) by using the above proposition. Indeed, if any solution of (4.20) can be represented as $u_t = u_0\big(Y_t^{-1}\big)$, where Y_t is the flow generated by $\tilde{A}_0$, then by the above proposition, we must have $\theta_t(\varphi_t) = \theta_0\big(Y_t^{-1}\big)$ (since $\theta_t(\varphi_t)|_{t=0} = \theta_0$), which gives us

$$\theta_t = \theta_0\big[Y_t^{-1}\big(\varphi_t^{-1}\big)\big] = \theta_0\big(X_t^{-1}\big).$$

That is to say, any solution of (4.8) is expressed as the composition of θ_0 and the inverse flow X_t^{-1}. However, since the divergence $\mathrm{div}(\tilde{A}_0)$ of $\tilde{A}_0$ is unbounded, it is difficult to get a meaningful uniqueness result for the equation (4.20), see [1, 2, 9].

5 Approximate differentiability of the flow generated by (1.1)

In this section we study the approximate differentiability of the stochastic flow X_t associated to the Stratonovich SDE (1.1) whose drift coefficient A_0 belongs to the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. To this end, we introduce some notations and results about maximal functions. For any bounded measurable subset $U \subset \mathbb{R}^d$ with positive Lebesgue measure $\mathcal{L}_d(U) > 0$, define the average of $f \in L^1_{loc}(\mathbb{R}^d)$ on U by

$$⨍_U f(x)\,\mathrm{d}x = \frac{1}{\mathcal{L}_d(U)} \int_U f(x)\,\mathrm{d}x.$$

Then for any $x \in \mathbb{R}^d$ and $R > 0$, the local maximal function $M_R f$ is defined by

$$M_R f(x) = \sup_{0<r\le R} ⨍_{B(x,r)} |f(y)|\,\mathrm{d}y,$$

where $B(x,r) = \{y \in \mathbb{R}^d : |y-x| \le r\}$. Here are some results regarding the maximal function (see [22]; for a proof of the second result, cf. the Appendix of [15]).

Lemma 5.1. (1) *For $R, \rho > 0$, there are C_d, $C_{d,\rho} > 0$ such that for all $f \in L^1_{loc}(\mathbb{R}^d)$, we have*

$$\int_{B(\rho)} M_R f(x)\,\mathrm{d}x \le C_{d,\rho} + C_d \int_{B(R+\rho)} |f(x)| \log(2 + |f(x)|)\,\mathrm{d}x$$

and for any $\alpha > 0$,

$$\mathcal{L}_d(x \in B(\rho) : M_R f(x) > \alpha) \le \frac{C_d}{\alpha} \int_{B(R+\rho)} |f(x)|\,\mathrm{d}x.$$

(2) *Let $f \in W^{1,1}_{loc}(\mathbb{R}^d)$. Then there is $C_d > 0$ (independent of f) and a negligible set $N \subset \mathbb{R}^d$, such that for all $x, y \in N^c$ with $|x-y| \le R$,*

$$|f(x) - f(y)| \le C_d |x-y| \big((M_R|\nabla f|)(x) + (M_R|\nabla f|)(y)\big).$$

We first prove the following result on the approximate differentiability of the regular Lagrangian flow generated by a Sobolev vector field b. This is an extension of Corollary 2.5 in [7] to the case where b has linear growth (see [7] Corollary 3.5 and [4] Remark 3.8 for more general case, but therein the divergence of b is assumed to be bounded on $\mathbb{R}^d$).

Proposition 5.2. *Assume that $b \in L^1\big([0,T], W^{1,1}_{loc}(\mathbb{R}^d, \mathbb{R}^d)\big)$ satisfying*

(i) $\frac{|b_t(x)|}{1+|x|} \in L^\infty([0,T]\times\mathbb{R}^d)$;

(ii) *for any* $R>0$, $\int_0^T \|\mathrm{div}(b_t)\|_{L^\infty(B(R))}\mathrm{d}t < +\infty$;

(iii) *for any* $R>0$, $\int_0^T\!\int_{B(R)} |\nabla b_t|\log(2+|\nabla b_t|)\,\mathrm{d}x\mathrm{d}t < +\infty$.

Let Y_t be the regular Lagrangian flow generated by b. Then for any $R>0$ and $\varepsilon>0$, there exists a Borel set $E\subset B(R)$ such that $\mathcal{L}_d(B(R)\setminus E)<\varepsilon$ and the restriction $Y_t|_E$ is a Lipschitz map for any $t\in[0,T]$.

In particular, for any $t\in[0,T]$, Y_t is approximately differentiable $\mathcal{L}_d$-a.e. in $\mathbb{R}^d$.

Proof. We follow the idea of the proof of Corollary 2.5 in [7]. For $0\le t\le T$, $0<r\le 2R$ and $x\in B(R)$, define

$$Q(t,x,r)=\fint_{B(x,r)}\log\left(\frac{|Y_t(x)-Y_t(y)|}{r}+1\right)\mathrm{d}y.$$

From Definition 2.4(1), it follows that for a.e. x and for all $r\in(0,2R]$, the map $t\to Q(t,x,r)$ is Lipschitz and

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t}(t,x,r)&\le \fint_{B(x,r)}\left|\frac{\mathrm{d}Y_t}{\mathrm{d}t}(x)-\frac{\mathrm{d}Y_t}{\mathrm{d}t}(y)\right|\cdot\big(|Y_t(x)-Y_t(y)|+r\big)^{-1}\mathrm{d}y\\ &=\fint_{B(x,r)}\frac{|b_t(Y_t(x))-b_t(Y_t(y))|}{|Y_t(x)-Y_t(y)|+r}\,\mathrm{d}y. \end{aligned}\tag{5.1}$$

By condition (i) and Gronwall's inequality, it is easy to show that

$$|Y_t(x)|\le(1+R)e^{CT},\quad \text{for all } x\in B(R),\ 0\le t\le T.$$

Therefore for a.e. $x\in B(R)$ and $y\in B(x,r)$, we have

$$|Y_t(x)-Y_t(y)|\le|Y_t(x)|+|Y_t(y)|\le(1+R)e^{CT}+(1+3R)e^{CT}=2(1+2R)e^{CT}=:\tilde R.$$

Since $(Y_t)_\#\mathcal{L}_d\ll\mathcal{L}_d$, we can apply Lemma 5.1(2) to get

$$|b_t(Y_t(x))-b_t(Y_t(y))|\le C_d|Y_t(x)-Y_t(y)|\cdot\big[(M_{\tilde R}|\nabla b_t|)(Y_t(x))+(M_{\tilde R}|\nabla b_t|)(Y_t(y))\big].$$

Substituting this estimate into (5.1) gives us

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t}(t,x,r)&\le\fint_{B(x,r)}C_d\big[(M_{\tilde R}|\nabla b_t|)(Y_t(x))+(M_{\tilde R}|\nabla b_t|)(Y_t(y))\big]\mathrm{d}y\\ &=C_d(M_{\tilde R}|\nabla b_t|)(Y_t(x))+C_d\fint_{B(x,r)}(M_{\tilde R}|\nabla b_t|)(Y_t(y))\,\mathrm{d}y.\end{aligned}$$

Integrating with respect to time, we obtain for all $t\in[0,T]$,

$$Q(t,x,r)\le\log 2+C_d\int_0^T(M_{\tilde R}|\nabla b_s|)(Y_s(x))\,\mathrm{d}s+C_d\int_0^T\fint_{B(x,r)}(M_{\tilde R}|\nabla b_s|)(Y_s(y))\,\mathrm{d}y\mathrm{d}s.$$

Let $\Phi(x)=\int_0^T(M_{\tilde R}|\nabla b_s|)(Y_s(x))\,\mathrm{d}s$, then by Fubini's theorem,

$$Q(t,x,r)\le\log 2+C_d\Phi(x)+C_d\fint_{B(x,r)}\Phi(y)\,\mathrm{d}y,\quad \text{for all } t\in[0,T].$$

Hence by the definition of the maximal function,

$$\sup_{0\le t\le T}\sup_{0<r\le 2R} Q(t,x,r) \le \log 2 + C_d\Phi(x) + C_d(M_{2R}\Phi)(x). \tag{5.2}$$

For η sufficiently small, we have

$$\begin{aligned}&\mathcal{L}_d\big(x\in B(R): \log 2 + C_d\Phi(x) + C_d(M_{2R}\Phi)(x) > 1/\eta\big)\\ &\quad\le \mathcal{L}_d\big(x\in B(R): C_d\Phi(x) > 1/(3\eta)\big) + \mathcal{L}_d\big(x\in B(R): C_d(M_{2R}\Phi)(x) > 1/(3\eta)\big).\end{aligned} \tag{5.3}$$

By Chebyshev's inequality,

$$\mathcal{L}_d\big(x\in B(R): C_d\Phi(x) > 1/(3\eta)\big) \le 3\eta C_d\int_{B(R)}\Phi(x)\,\mathrm{d}x.$$

Using Lemma 5.1(1), we have

$$\mathcal{L}_d\big(x\in B(R): C_d(M_{2R}\Phi)(x) > 1/(3\eta)\big) \le 3\eta C_d C_d'\int_{B(3R)}\Phi(x)\,\mathrm{d}x.$$

Substituting these two estimates into (5.3) and by the definition of $\Phi(x)$, we obtain

$$\begin{aligned}I &:= \mathcal{L}_d\big(x\in B(R): \log 2 + C_d\Phi(x) + C_d(M_{2R}\Phi)(x) > 1/\eta\big)\\ &\le 3\eta C_d(1+C_d')\int_{B(3R)}\Phi(x)\,\mathrm{d}x\\ &= 3\eta C_d(1+C_d')\int_0^T\int_{B(3R)}(M_{\tilde R}|\nabla b_t|)(Y_t(x))\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Using the density $\tilde\rho_t$ of the flow Y_t, we get

$$I \le 3\eta C_d(1+C_d')\int_0^T\int_{Y_t(B(3R))}(M_{\tilde R}|\nabla b_t|)(y)\tilde\rho_t(y)\,\mathrm{d}y\mathrm{d}t.$$

In view of the expression of $\tilde\rho_t$ given in Remark 2.6, for any $x\in B(3R)$ and $t\in[0,T]$,

$$\tilde\rho_t(Y_t(x)) = \exp\bigg(-\int_0^t \mathrm{div}(b_s)(Y_s(x))\,\mathrm{d}s\bigg) \le \exp\bigg(\int_0^T \|\mathrm{div}(b_s)\|_{L^\infty(B(R_1))}\mathrm{d}s\bigg) =: L,$$

where $R_1 = (1+3R)e^{CT}$. Hence by Lemma 5.1(1),

$$\begin{aligned}I &\le 3\eta C_d(1+C_d')L\int_0^T\int_{B(R_1)}(M_{\tilde R}|\nabla b_t|)(y)\,\mathrm{d}y\mathrm{d}t\\ &\le 3\eta C_d(1+C_d')L\int_0^T\bigg[C_{d,R_1} + C_d''\int_{B(R_1+\tilde R)}|\nabla b_t|\log(2+|\nabla b_t|)\,\mathrm{d}y\bigg]\mathrm{d}t\\ &=: \eta L_1.\end{aligned}$$

Now for any $\varepsilon>0$, let $\eta=\varepsilon/L_1$, then by (5.2) and the definition of I, we have

$$\mathcal{L}_d\bigg(x\in B(R): \sup_{0\le t\le T}\sup_{0<r\le 2R} Q(t,x,r) > \frac{L_1}{\varepsilon}\bigg) \le I \le \frac{\varepsilon}{L_1}\cdot L_1 = \varepsilon.$$

Let

$$E = \bigg\{x\in B(R): \sup_{0\le t\le T}\sup_{0<r\le 2R} Q(t,x,r) \le \frac{L_1}{\varepsilon}\bigg\},$$

then $\mathcal{L}_d(B(R)\setminus E)\le\varepsilon$ and for any $x\in E$, $0\le t\le T$ and $0<r\le 2R$, the definition of $Q(t,x,r)$ leads to

$$\fint_{B(x,r)}\log\left(\frac{|Y_t(x)-Y_t(y)|}{r}+1\right)\mathrm{d}y\le\frac{L_1}{\varepsilon}. \tag{5.4}$$

Now fix any $x,y\in E$ and let $r=|x-y|$ which is less than $2R$. We have by the triangular inequality,

$$\begin{aligned}\log\left(\frac{|Y_t(x)-Y_t(y)|}{r}+1\right)&\le\log\left(\frac{|Y_t(x)-Y_t(z)|+|Y_t(z)-Y_t(y)|}{r}+1\right)\\&\le\log\left(\frac{|Y_t(x)-Y_t(z)|}{r}+1\right)+\log\left(\frac{|Y_t(z)-Y_t(y)|}{r}+1\right),\end{aligned}$$

therefore by (5.4),

$$\begin{aligned}&\log\left(\frac{|Y_t(x)-Y_t(y)|}{r}+1\right)\\&\quad=\fint_{B(x,r)\cap B(y,r)}\log\left(\frac{|Y_t(x)-Y_t(y)|}{r}+1\right)\mathrm{d}z\\&\quad\le\fint_{B(x,r)\cap B(y,r)}\left[\log\left(\frac{|Y_t(x)-Y_t(z)|}{r}+1\right)+\log\left(\frac{|Y_t(z)-Y_t(y)|}{r}+1\right)\right]\mathrm{d}z\\&\quad\le\tilde C_d\fint_{B(x,r)}\log\left(\frac{|Y_t(x)-Y_t(z)|}{r}+1\right)\mathrm{d}z+\tilde C_d\fint_{B(y,r)}\log\left(\frac{|Y_t(z)-Y_t(y)|}{r}+1\right)\mathrm{d}z\\&\quad\le 2\tilde C_d\cdot\frac{L_1}{\varepsilon},\end{aligned}$$

where $\tilde C_d=\mathcal{L}_d(B(x,r))/\mathcal{L}_d(B(x,r)\cap B(y,r))$ only depends on the dimension d. Therefore

$$|Y_t(x)-Y_t(y)|\le re^{2\tilde C_dL_1/\varepsilon}=|x-y|e^{2\tilde C_dL_1/\varepsilon}$$

which holds for all $x,y\in E$. Hence $\mathrm{Lip}(Y_t|_E)\le e^{2\tilde C_dL_1/\varepsilon}$. □

Now we can prove the main result of this section.

Theorem 5.3. *Assume that $A_1,\cdots,A_m\in C_b^{3+\delta}(\mathbb{R}^d,\mathbb{R}^d)$ and that $A_0\in W^{1,1}_{loc}(\mathbb{R}^d,\mathbb{R}^d)$ satisfies*

(1) *A_0 has sublinear growth;*

(2) *$\mathrm{div}(A_0)$ is locally bounded on $\mathbb{R}^d$;*

(3) *for any $R>0$, $\int_{B(R)}\|\nabla A_0\|\log(2+\|\nabla A_0\|)\,\mathrm{d}x<+\infty$.*

Then for a.s. $w\in\Omega_0$, for any $R>0$ and $\delta>0$, there exists a Borel set $E\subset B(R)$ such that $\mathcal{L}_d(B(R)\setminus E)<\delta$ and the restriction of the flow X_t to E is a Lipschitz map for any $t\in[0,T]$. In particular, X_t is approximately differentiable $\mathcal{L}_d$-a.e. in $\mathbb{R}^d$ for any $t\in[0,T]$.

Proof. Since $X_t=\varphi_t(Y_t)$ and for a.s. $w\in\Omega_0$, the map $\varphi_t:\mathbb{R}^d\to\mathbb{R}^d$ is a C^2-diffeomorphism on $\mathbb{R}^d$, we only have to prove the result for the solution Y_t of the random ODE (3.2). Now we check that $\tilde A_0$ satisfies the conditions given in Proposition 5.2. First by the definition of $\tilde A_0(t,\cdot)$ and the conditions on A_0, it is clear that $\tilde A_0(t,\cdot)\in L^1([0,T],L^1_{loc}(\mathbb{R}^d,\mathbb{R}^d))$. Moreover

$$\nabla\tilde A_0(t,x)=(\nabla K_t(x))A_0(\varphi_t(x))+K_t(x)(\nabla A_0)(\varphi_t(x))J_t(x),$$

hence

$$\|\nabla\tilde A_0(t,x)\|\le\|\nabla K_t(x)\|\cdot|A_0(\varphi_t(x))|+\|K_t(x)\|\cdot\|J_t(x)\|\cdot\|(\nabla A_0)(\varphi_t(x))\|. \tag{5.5}$$

The terms $\|\nabla K_t(x)\|$, $\|K_t(x)\|$ and $\|J_t(x)\|$ are bounded on $[0,T]\times B(R)$. By Lemma 3.1 and the fact that A_0 has sublinear growth, it is easy to show that $|A_0(\varphi_t(x))|$ has an upper bound on $[0,T]\times B(R)$. As for the last term in (5.5), noticing that $L:=\cup_{0\le t\le T}\varphi_t(B(R))$ is a bounded subset, we have

$$\begin{aligned}\int_0^T\int_{B(R)}\|\nabla\tilde A_0(t,x)\|\,\mathrm{d}x\mathrm{d}t &\le C_{T,R}+C'_{T,R}\int_0^T\int_{B(R)}\|(\nabla A_0)(\varphi_t)\|\,\mathrm{d}x\mathrm{d}t\\ &= C_{T,R}+C'_{T,R}\int_0^T\int_{\varphi_t(B(R))}\|\nabla A_0\|\cdot|\det(K_t)(\varphi_t^{-1})|\,\mathrm{d}x\mathrm{d}t\\ &\le C_{T,R}+C''_{T,R}\int_L\|\nabla A_0\|\,\mathrm{d}x<+\infty,\end{aligned}$$

where the last inequality follows from the boundedness of $|\det(K_t)|$ on $[0,T]\times B(R)$. Hence $\tilde A_0\in L^1([0,T],W^{1,1}_{loc}(\mathbb{R}^d,\mathbb{R}^d))$.

By Lemma 3.1, the condition (i) in Proposition 5.2 is easily checked for $\tilde A_0$. The second condition (ii) can be verified by using the equality in Lemma 2.2(2), as we have done at the end of the proof of Proposition 3.2.

Now we check that $\tilde A_0$ satisfies the condition in Proposition 5.2(iii). Again by (5.5) and the above discussions, we have $\|\nabla\tilde A_0(t)\|\le C_{T,R}(1+\|(\nabla A_0)\circ\varphi_t\|)$. Therefore by the simple inequality $\log(1+s)\le s$ for all $s\ge 0$, we have

$$\begin{aligned}\log\big(2+\|\nabla\tilde A_0(t)\|\big) &\le \log\big[(2+C_{T,R})(2+\|(\nabla A_0)\circ\varphi_t\|)\big]\\ &\le (1+C_{T,R})+\log\big(2+\|(\nabla A_0)\circ\varphi_t\|\big).\end{aligned}$$

As a result,

$$\begin{aligned}&\|\nabla\tilde A_0(t)\|\log\big(2+\|\nabla\tilde A_0(t)\|\big)\\ &\quad\le C_{T,R}(1+C_{T,R})\big(1+\|(\nabla A_0)\circ\varphi_t\|\big)\big[1+\log\big(2+\|(\nabla A_0)\circ\varphi_t\|\big)\big]\\ &\quad\le C_{T,R}(1+C_{T,R})\big[2\big(1+\|(\nabla A_0)\circ\varphi_t\|\big)+\|(\nabla A_0)\circ\varphi_t\|\log\big(2+\|(\nabla A_0)\circ\varphi_t\|\big)\big].\end{aligned}\tag{5.6}$$

Again by the fact that $L:=\cup_{0\le t\le T}\varphi_t(B(R))$ is bounded for any $R>0$, we have by the condition (3) that

$$\begin{aligned}&\int_0^T\int_{B(R)}\|(\nabla A_0)\circ\varphi_t\|\log\big(2+\|(\nabla A_0)\circ\varphi_t\|\big)\mathrm{d}x\mathrm{d}t\\ &\quad\le\int_0^T\int_{\varphi_t(B(R))}\|\nabla A_0\|\log\big(2+\|\nabla A_0\|\big)|\det(K_t)\circ\varphi_t^{-1}|\,\mathrm{d}y\mathrm{d}t\\ &\quad\le C'_{T,R}T\int_L\|\nabla A_0\|\log\big(2+\|\nabla A_0\|\big)\big]\mathrm{d}y<+\infty,\end{aligned}$$

where $C'_{T,R}=\sup\{|\det(K_t(x))|:(t,x)\in[0,T]\times B(R)\}<+\infty$. This and (5.6) clearly imply that

$$\int_0^T\int_{B(R)}\|\nabla\tilde A_0(t,x)\|\log\big(2+\|\nabla\tilde A_0(t,x)\|\big)\mathrm{d}x\mathrm{d}t<+\infty.$$

The condition (iii) in Proposition 5.2 is verified and the proof is complete. □

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